

Stability issues with baryons in AdS/CFT

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ABSTRACT: We consider baryon vertices within the gauge/gravity correspondence for a class of curved backgrounds. The holographic description based on the $\mathcal{N} = 4$ SYM theory for $SU(N)$ allows classical solutions representing bound states of k -quarks with k less than or equal to N . We construct the corresponding classical configurations and perform a stability analysis. We present the details for the theory at the conformal point and at finite temperature and show that there is a critical value of k , below which there is instability. This may also arise when the baryon reaches a critical size. We also extend our treatment to magnetically charged baryon vertices.

KEYWORDS: Gauge-gravity correspondence, D-branes, AdS-CFT Correspondence, Bosonic Strings.

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1. Introduction

The basic construction in use to understand baryons in the context of the gauge/gravity correspondence [1] involves N heavy external quarks on the boundary of AdS_5 at the end points of N fundamental strings terminating at a vertex on the interior of AdS_5 . Each string contributes a unit of charge, so that for the total charge to be conserved at the vertex, it was proposed in [2] that a $D5$ -brane wrapped on the internal S^5 -sphere should be placed there. Due to its presence, a coupling via a Wess-Zumino term to the Dirac-Born-Infeld (DBI) electric field strength ensures that the total charge indeed is preserved. This idea was actually applied in calculating the classical solution corresponding to such a configuration in [3, 4] and involved generalizations of techniques developed for the heavy quark-antiquark system at the conformal point of $\mathcal{N} = 4$ SYM for $SU(N)$ in [5].

In recent works [6–8] we have extensively studied within the gauge/gravity correspondence, the stability of interaction potentials between a heavy quark and anti-quark in

mesons, as well as between a quark and monopole, for a general class of backgrounds. A general outcome of these works is that a stability analysis is very important in establishing the physical parameter space of the theory which is typically reduced in comparison with that allowed from simply the requirement that the classical solutions exist. A second lesson is that in cases of multibranch solutions the point where instability begins does not always coincide with the point where a solution bifurcates. It seems that this feature requires the coexistence in the classical configuration of fundamental as well as of solitonic objects. In particular, this was the case of the potential for the quark-monopole pair in which a string junction was used that requires precisely a fundamental-, a D - and a dyon-string.

The case of baryons we focus in this paper presents a new challenge and requires special care. In particular, at a technical level it requires a multistring configuration with N strings. In addition, the vertex where these meet in the bulk of AdS_5 is not a mathematical point, as in the case of a string junction, but it is a place where a solitonic object, the $D5$ -brane wrapped on the internal S^5 , is placed. A further challenge, that initially prompted our analysis, is that there have been configurations with $5N/8 < k < N$ quarks [3, 4] that are perfect as classical solutions to the equations of motion, besides the expected ones with $k = N$. This fact is against physical intuition and experimental results that, bound states of quarks and antiquarks are singlets of the gauge group. Of course in the case at hand we deal with a maximally supersymmetric theory, but one would like to show that a better bound for k/N can be found, if not $k = N$ exactly. It is the purpose of the present paper to address these and related issues.

The organization of this paper is as follows: In section 2 we formulate baryons for a general class of backgrounds. In particular, we derive general formulas for the baryon binding energy for the case of quarks uniformly distributed on a spherical shell when forming the baryon. In section 3 we perform a stability analysis under small fluctuations and derive their equations of motion as well as the boundary and matching conditions at the boundary of AdS_5 and at the baryon vertex. In section 4 we present examples of baryon configurations within the gauge/gravity correspondence, using the conformal and finite temperature backgrounds. We will show that stability restricts the value of k to be larger than a critical value even in the conformal case (a higher than $5N/8$ value). Also even a part of the energetically favorable branch of multi-branched potentials should be disregarded as unstable. In section 5 we extend our treatment to the case of magnetically charged baryon vertices and show that stability requires an upper bound for the instanton number, as compared to the 't Hooft coupling, associated with a self-dual gauge field. Finally, in section 6, we summarize our results and discuss some future directions.

2. The classical solution

In this section we develop and present the general setup of the gauge/gravity calculation of the potential energy of a baryon consisting of heavy static quarks. Since the underlying dual theory is $\mathcal{N} = 4$ SYM for $SU(N)$, these computations involve N strings as well a $D5$ -brane wrapped on the internal S^5 -sphere. Certain elements of the computation will be based on the well known techniques used for the calculation of the potential of a heavy

quark-antiquark pair [5] as generalized beyond the conformal point in [9], as well as the concept of the baryon vertex [2]. Our classical construction will essentially generalize that for the conformal case in [3, 4] to a more general class of backgrounds.

We consider diagonal metrics of Lorentzian signature of the form

$$ds^2 = G_{tt}dt^2 + G_{xx}(dx^2 + dy^2 + dz^2) + G_{uu}du^2 + R^2d\Omega_5^2, \quad (2.1)$$

where x, y and z denote cyclic coordinates and u denotes the radial direction playing the rôle of an energy scale in the dual gauge theory. It extends from the UV at $u \rightarrow \infty$ down to the IR at some minimum value u_{\min} determined by the geometry.

It is convenient to introduce the functions

$$f(u) = -G_{tt}G_{xx}, \quad g(u) = -G_{tt}G_{uu}, \quad h(u) = G_{yy}G_{uu}. \quad (2.2)$$

In the conformal limit we can approximate the metric by that for $AdS^5 \times S^5$ with radii ($\alpha' = 1$) $R = (4\pi g_s N)^{1/4}$, with g_s being the string coupling. In this limit the leading order expressions are

$$f(u) \simeq u^4, \quad g(u) \simeq 1, \quad h(u) \simeq 1. \quad (2.3)$$

A baryon is by definition a bound state of N -quarks which form the completely antisymmetric representation of $SU(N)$. However, within the gauge/gravity correspondence, we can in general construct a bound state of k -quarks with $k < N$. This bound state consists of a $D5$ -brane located in the bulk and k -external quarks at the boundary of AdS_5 . Each quark is connected with a string to the $D5$ -brane. There are also $N - k$ straight strings that go from the $D5$ -brane straight up at u_{\min} . This description is depicted in figure 1. We will take $N, k \gg 1$, such that the ratio k/N is held finite.

At this point we have to choose a profile for the quark distribution inside the baryon. We expect a radially symmetric distribution and for simplicity we take it to be a uniform distribution on a spherical shell of radius L . We will briefly comment on other choices and on how the preferable distribution could be determined in the last section. Also note the entire description is valid only if the endpoints of the N -strings are uniformly distributed on the internal S^5 -sphere so that the latter is not deformed and the probe approximation still holds (for a discussion on this point see [4]). Such a choice completely breaks supersymmetry [10, 11].

Within the gauge/gravity correspondence, the binding potential energy of the baryon is given by

$$e^{-iET} = e^{iS_{\text{cl}}}, \quad (2.4)$$

where S_{cl} is the classical action of the above configuration.

This action consists of three terms, a Nambu-Goto action for each of the N -strings, a Dirac-Born-Infeld action for the $D5$ -brane and a Wess-Zumino term between the N -strings

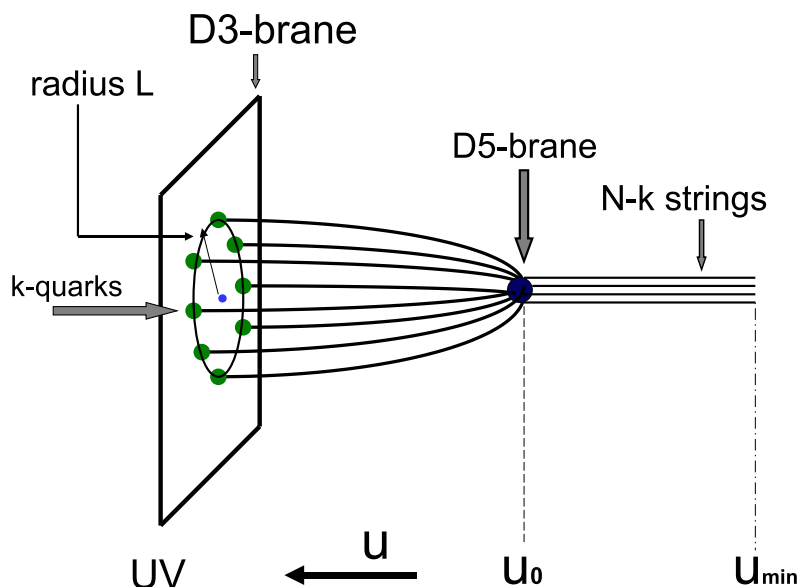


Figure 1: A baryon configuration with k -external quarks placed on a spherical shell of radius L at the boundary of AdS space, each connected to the $D5$ -brane which is wrapped on an S^5 located at $u = u_0$ and $N - k$ straight strings ending at u_{\min} .

and the $D5$ -brane

$$\begin{aligned}
 S_{NG}[C] &= \frac{1}{2\pi} \int_C d\tau d\sigma \sqrt{-\det g_{\alpha\beta}}, \\
 S_{D5} &= \frac{1}{(2\pi)^5 g_s} \int_M d^6x \sqrt{-h_{ij} - 2\pi F_{ij}}, \\
 S_{WZ} &= \frac{1}{(2\pi)^5 g_s} \int_M d^6x C^{(4)} \wedge F, \quad M = \mathbb{R} \times S^5,
 \end{aligned}
 \tag{2.5}$$

where $g_{\alpha\beta}, h_{ij}, F_{ij}$ are the induced metrics for the string, $D5$ -brane and Born-Infeld field strength, respectively. We will consider the simplest case where the DBI-action is just given by its induced metric and the presence of the Wess-Zumino term acts as a charge conservation term [2]. However, there are cases of non-trivial F_{ij} [12] to which we will return later in section 5. We have not also included the contribution of the antisymmetric NS-NS two-form due to the fact that it is not present in our examples.

We first fix reparametrization invariance for each string by choosing

$$t = \tau, \quad u = \sigma. \tag{2.6}$$

For static solutions we consider the embedding of the S^2 -sphere on the $D3$ -brane in spherical coordinates (r, θ, ϕ)

$$r = r(u), \quad \phi, \theta = \text{const.}, \quad S^5\text{-angles} = \text{const.}, \tag{2.7}$$

supplemented by the boundary condition

$$u(L) = \infty. \tag{2.8}$$

Then, the Nambu-Goto action reads

$$S = -\frac{T}{2\pi} \int du \sqrt{g(u) + f(u)r'^2}, \quad (2.9)$$

where T denotes time and the prime denotes a derivative with respect to u . From the Euler-Lagrange equations of motion we obtain

$$\frac{fr'_{\text{cl}}}{\sqrt{g + fr'^2_{\text{cl}}}} = f_1^{1/2} \quad \implies \quad r'_{\text{cl}} = \frac{\sqrt{f_1 F}}{f}, \quad (2.10)$$

where u_1 is the value of u at the turning point of each string, $f_1 \equiv f(u_1)$, $f_0 \equiv f(u_0)$ and

$$F = \frac{gf}{f - f_1}. \quad (2.11)$$

The $N - k$ straight strings which extend from the baryon vertex to $u = u_{\text{min}}$ are straight, since $r = \text{const.}$ is always a solution of the equations of motion (with $f_1 = 0$). In the examples to follow $u_{\text{min}} \leq u_1 \leq u_0$ and $f(u)$ is an increasing function. Integrating (2.10), we can express the radius of the spherical shell as

$$L = \sqrt{f_1} \int_{u_0}^{\infty} du \frac{\sqrt{F}}{f}. \quad (2.12)$$

Next we fix the reparametrization invariance for the wrapped $D5$ -brane by choosing

$$t = \tau, \quad \theta_a = \sigma_\alpha, \quad \alpha = 1, 2, \dots, 5. \quad (2.13)$$

For the ansatz given above, the DBI action reads

$$S_{D5} = \frac{TNR}{8\pi} \sqrt{-G_{tt}} \Big|_{u=u_0}, \quad (2.14)$$

where we have also used that the volume of the unit- S^5 is π^3 .

Finally, inserting the solution for r'_{cl} into (2.9), subtracting the divergent self-energy contribution of disconnected worldsheets and adding the energy of the $D5$ -brane (2.14), we write the binding energy of the baryon as

$$E = \frac{k}{2\pi} \left\{ \int_{u_0}^{\infty} du \sqrt{F} - \int_{u_{\text{min}}}^{\infty} du \sqrt{g} + \frac{1-a}{a} \int_{u_{\text{min}}}^{u_0} du \sqrt{g} + \frac{R\sqrt{-G_{tt}}}{4a} \Big|_{u=u_0} \right\}, \quad (2.15)$$

where

$$a \equiv \frac{k}{N}, \quad 0 < a \leq 1. \quad (2.16)$$

The expressions for the length and the energy, (2.12) and (2.15) depend on the arbitrary parameter u_1 which should be expressed in terms of the baryon vertex position u_0 . The most convenient way to find this is to impose that the net force at the baryon vertex is zero. Following an analogous procedure to that for the string junction in [8], we temporarily change gauge to $r = \sigma$ and from the boundary terms arising in deriving the classical

equations of motion we find that for a uniform distribution of the quarks on a spherical shell, we get the relation (reducing to that in [3] for the conformal case)

$$\begin{aligned} \cos \Theta &= \frac{1-a}{a} + \frac{R}{4a\sqrt{g}} \partial_u \sqrt{-G_{tt}} \Big|_{u=u_0}, \\ \cos \Theta &= \sqrt{1 - f_1/f_0}, \end{aligned} \tag{2.17}$$

where Θ is the angle between each of the k -strings and the u -axis at the baryon vertex and which determines u_1 in terms of u_0 . The above relation can also be interpreted as the no-force condition for its u -component [4]. The term on the left hand side of (2.17) is due to the strings attached to the boundary at the UV. The force they exert on the baryon vertex is balanced by the forces exerted on that by the straight strings and the tendency of the $D5$ -brane to move towards u_{\min} , represented by the first term and the second term on the right hand side of (2.17) (the second term is indeed positive in our examples). Hence, if a becomes small enough we expect that this condition might not be possible to satisfy. As we will see in the examples to follow, (2.17) has a solution for a parametric region of (a, u_0) . However, in order to isolate parametric regions of physical interest a stability analysis of the classical solution should be performed, which, as we will see, further restricts the allowed region.

3. Stability analysis

We now turn to the stability analysis of the above classical configuration, aiming at isolating the physical parametric regions. This involves string fluctuations which fall into the general results analyzed in [6, 7] for a single string. However, the quantitative results are quite different, due to the relative orientation of the strings, being distributed on a sphere, and the different boundary conditions that will be imposed here.

3.1 Small fluctuations

We parametrize the string fluctuations about the classical solution, by perturbing the embedding according to

$$r = r_{\text{cl}}(u) + \delta r(t, u), \quad \theta = \theta_0 + \delta \theta(t, u), \quad \phi = \phi_0 + \delta \phi(t, u). \tag{3.1}$$

We have kept the gauge choice (2.6) unperturbed by using worldsheet reparametrization invariance. We then calculate the Nambu-Goto action for this ansatz and we expand it in powers of the fluctuations. The resulting expansion for the quadratic fluctuations for each one of the k -strings can be written as

$$\begin{aligned} S_2 = -\frac{1}{2\pi} \int dt du & \left[\frac{gf}{2F^{3/2}} \delta r'^2 - \frac{h}{2F^{1/2}} \delta \dot{r}^2 + \frac{fr_{\text{cl}}^2}{2F^{1/2}} \delta \theta'^2 - \frac{hr_{\text{cl}}^2 F^{1/2}}{2g} \delta \dot{\theta}^2 \right. \\ & \left. + \frac{fr_{\text{cl}}^2 \sin^2 \theta_0}{2F^{1/2}} \delta \phi'^2 - \frac{hr_{\text{cl}}^2 \sin^2 \theta_0 F^{1/2}}{2g} \delta \dot{\phi}^2 \right]. \end{aligned} \tag{3.2}$$

Using the Euler-Lagrange equations and the ansatz

$$\delta x^\mu(t, u) = \delta x^\mu(u) e^{-i\omega t}, \quad x^\mu = r, \theta, \phi, \tag{3.3}$$

we find for each of the k -strings the linearized equations for the longitudinal and transverse fluctuations

$$\begin{aligned} \left[\frac{d}{du} \left(\frac{gf}{F^{3/2}} \frac{d}{du} \right) + \omega^2 \frac{h}{F^{1/2}} \right] \delta r &= 0, \\ \left[\frac{d}{du} \left(\frac{fr_{\text{cl}}^2}{F^{1/2}} \frac{d}{du} \right) + \omega^2 \frac{hr_{\text{cl}}^2 F^{1/2}}{g} \right] \delta \theta &= 0, \\ \left[\frac{d}{du} \left(\frac{fr_{\text{cl}}^2}{F^{1/2}} \frac{d}{du} \right) + \omega^2 \frac{hr_{\text{cl}}^2 F^{1/2}}{g} \right] \delta \phi &= 0. \end{aligned} \tag{3.4}$$

For the straight strings the expressions for the action and the equations of motion were given in [8] and will not be repeated here. As discussed extensively in [6, 8] instabilities are indicated by the presence of a normalizable zero-mode which can exist for specific values of u_0 , denoted by u_{0c} . Such values are in general different than possible values of u_0 at which the length reaches an extremum, denoted by u_{0m} and found by solving $L'(u_0) = 0$. Our aim is to find for the zero mode the critical curve in the parametric space of a and u_0 which separates the stable from the unstable regions.

Finally, we turn to the fluctuations of the $D5$ -brane and perturb the embedding according to

$$x^\mu = \delta x^\mu(t, \theta_\alpha), \quad u = u_0, \quad x^\mu = x, y, z, \tag{3.5}$$

leaving the position of the $D5$ -brane at $u = u_0$ intact due to the gauge choice $u = \sigma$ for the strings. The second order, in the fluctuations, term in the expansion of the $D5$ -brane action is

$$S = \frac{NR}{8\pi^4} \int dt d\Omega_5 \sqrt{\gamma} \sqrt{-G_{tt}} \left\{ 1 + \frac{G_{\mu\nu}}{2} \gamma^{\alpha\beta} \partial_\alpha \delta x^\mu \partial_\beta \delta x^\nu + \frac{G_{\mu\nu}}{2G_{tt}} \delta \dot{x}^\mu \delta \dot{x}^\nu \right\}, \tag{3.6}$$

where $\gamma_{\alpha\beta}$ is the metric of S^5 and the action is calculated at $u = u_0$. The subscripts α, μ refer to the angles of S^5 and to the x, y, z coordinates, respectively. We expand the fluctuations in terms of the spherical harmonics of the S^5 -sphere Ψ_ℓ , satisfying the eigenvalue eq. $\nabla_\gamma^2 \Psi_\ell = -\ell(\ell + 4)\Psi_\ell$, $\ell = 0, 1, 2, \dots$, as

$$\delta x^\mu(t, \theta_\alpha) = \delta x^\mu(t) \Psi_\ell(\theta_\alpha). \tag{3.7}$$

Then, from the Euler-Lagrange equations for the action we find that

$$\frac{d^2 \delta x^\mu}{dt^2} + \Omega_\ell^2 \delta x^\mu = 0, \quad \Omega_\ell^2 = -G_{tt}(u_0) \ell(\ell + 4) > 0. \tag{3.8}$$

Thus the classical solution is stable under the δx^μ fluctuations. Note that, there are no boundary conditions for these fluctuations, the reason being that the $\mathbb{R} \times S^5$ has no boundary.

3.2 Boundary and matching conditions

To fully specify our eigenvalue problem, we must impose boundary conditions on the string fluctuations in the UV limit, $u \rightarrow \infty$ and in the IR limit, $u = u_{\min}$, as well as matching conditions at the baryon vertex, i.e. $u = u_0$. The boundary condition at the UV are chosen so that the quarks are kept fixed, that is

$$\Phi(u) = 0, \quad \text{as } u \rightarrow \infty, \quad (3.9)$$

where Φ is any of the fluctuations of any of the k -strings. At $u = u_{\min}$ we demand finiteness of the solution and its u -derivative for the $N - k$ straight strings that extend in there, so that the perturbative approach is valid. It turns out that for the zero-mode they completely decouple from the analysis, as in the case of the quark-monopole potential in [8]. Since we consider a uniform distribution of quarks on a spherical shell, the $D5$ -brane will be located by symmetry at $x = y = z = 0$. For each one of the k -strings, δr is its longitudinal fluctuation and $\delta\theta, \delta\phi$, are its transverse ones.

For the longitudinal fluctuations δr , it is more convenient to work out the necessary boundary condition in the gauge $r = \sigma$, instead of $u = \sigma$ and then translate the result in the latter gauge where the equations of motion are simpler. In that spirit, following also an analogous treatment in [6, 8], the gauge change is effectively done by the coordinate transformation

$$u = \bar{u} + \delta u(t, u), \quad \delta u(t, u) = -\frac{\delta r(t, u)}{r'_{\text{cl}}}. \quad (3.10)$$

Hence, the boundary condition at the baryon vertex can be found, if we temporarily change gauge to $r = \sigma$ and apply small fluctuations for the \bar{u} -coordinate. Demanding a well behaved boundary problem for δu -fluctuations we obtain

$$\left. \frac{d\delta u}{dr} \right|_{u=u_0} = 0. \quad (3.11)$$

From (2.10), (3.10) and (3.11) we find that the δr -fluctuations, for each of the k -strings, satisfy the following boundary condition

$$u = u_0 : \quad 2(f - f_1)\delta r' + \delta r \left(2f' - \frac{f'}{f}f_1 - \frac{g'}{g}(f - f_1) \right) = 0. \quad (3.12)$$

This is an equation for u_0 in terms of a and the parameters of the specific problem. We will denote its solution, whenever it exists, by u_{0c} . We will not assume continuity relations between the fluctuations of the $D5$ -brane and the N -strings.

In general, the value of u_{0c} does not coincide with that corresponding to a possible extremum of the radius $L(u_0)$, denoted by u_{0m} . For the quark-antiquark potential these values coincide [6, 7], but they do not do so for the case of the quark-monopole potential [8]. In our case using (2.12), (2.15) and (2.17) we find that

$$\begin{aligned} \frac{dE}{du_0} &= \frac{k\sqrt{f_1}}{2\pi} \frac{dL}{du_0}, \\ \frac{dL}{du_0} &= -\frac{\sqrt{f_1 F}}{f} + \frac{\partial u_1}{\partial u_0} \frac{f'_1}{2\sqrt{f_1}} \int_{u_0}^{\infty} du \frac{\sqrt{g f}}{(f - f_1)^{3/2}}, \end{aligned} \quad (3.13)$$

where from (2.17) we compute that

$$f_1' \frac{\partial u_1}{\partial u_0} = \sin^2 \Theta f_0' - \frac{R}{2a} f_0 \cos \Theta \partial_u \left(\frac{\partial_u \sqrt{-G_{tt}}}{\sqrt{g}} \right) \Big|_{u=u_0}. \quad (3.14)$$

A necessary condition for u_{0m} to exist is that the right hand side of (3.14) is positive, which is indeed satisfied in the non-extremal $D3$ -branes example we will examine in the next section. Also the extrema of $L(u_0)$ and $E(u_0)$ at $u_0 = u_{0m}$ coincide as in the case of the quark-antiquark and quark-monopole potential energy interactions shown in [7] and [8], respectively. Similarly to [9], we can easily show that

$$\begin{aligned} \frac{dE}{dL} &= \frac{k}{2\pi} \sqrt{f_1}, \\ \frac{d^2E}{dL^2} &= \frac{k}{4\pi} \frac{f_1'}{\sqrt{f_1}} \frac{1}{L'(u_0)} \frac{\partial u_1}{\partial u_0}. \end{aligned} \quad (3.15)$$

Since $dE/dL > 0$ the force is manifestly attractive. On the other hand its magnitude may decrease or increase with distance depending on the sign of the right had side of the expression for d^2E/dL^2 . Note that this is not necessarily in conflict with gauge theory expectations since there is no analog for the binding energy of baryons of the concavity condition of the heavy quark-antiquark pair found in [18]. In addition, having assumed that the quarks, the baryon consists of, are located on a spherical shell has an inherit degree of arbitrariness that is reflected in the details of the expressions we have derived.

3.3 Zero modes

We will now study the zero mode problem for a spherical shell distribution and find the critical curve in the parametric region of (a, u_0) .

The longitudinal and transverse fluctuations of each string, satisfying already the conditions in the UV, are

$$\begin{aligned} \delta r &= A J(u), & \delta \theta &= B K(u), & \delta \phi &= C K(u), \\ J(u) &= \int_u^\infty du \frac{\sqrt{gf}}{(f-f_1)^{3/2}}, & K(u) &= \int_u^\infty \frac{du}{fr_{cl}^2} \sqrt{\frac{gf}{f-f_1}}, \end{aligned} \quad (3.16)$$

where A, B and C are integration constants. Using (2.3) and (2.10) we see that $K(u)$ diverges, thus we should choose $B = C = 0$. Therefore, the transverse fluctuations are stable as in the cases of the quark-antiquark and quark-monopole potentials [6, 8].

For the longitudinal fluctuations we should substitute the above expression for δr into (3.12) and obtain a transcendental equation that defines the critical curve separating the stable from the unstable regions.

4. Examples

In this section we will first review the behavior of the baryon potential emerging in the conformal case [3, 4] and examine its stability behavior. Then we will present the analogous analysis for the finite temperature case.

4.1 The conformal case

4.1.1 Classical solution

At first we will consider the conformal case, in which the metric of $AdS_5 \times S^5$ has the form of (2.1), with

$$-G_{tt} = G_{xx} = G_{uu}^{-1} = \frac{u^2}{R^2}. \quad (4.1)$$

The radius and the energy corresponding to a uniform distribution of quarks on a spherical shell are given in terms of the position of the $D5$ -brane u_0 and the turning point u_1 of each string

$$L = \frac{R^2 u_1^2}{3u_0^3} \mathcal{I}, \quad E = \frac{ku_0}{2\pi} \left(-\mathcal{J} + \frac{5-4a}{4a} \right), \quad (4.2)$$

with

$$\mathcal{I} = {}_2F_1 \left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}; \frac{u_1^4}{u_0^4} \right), \quad \mathcal{J} = {}_2F_1 \left(-\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; \frac{u_1^4}{u_0^4} \right) \quad (4.3)$$

and where ${}_2F_1(a, b, c; z)$ is the hypergeometric function and $u_0 \geq u_1 \geq 0$. From (2.17) we find that in this case the no-force condition on the u -axis yields

$$u_1 = u_0(1 - \lambda^2)^{1/4}, \quad \lambda = \frac{5-4a}{4a}. \quad (4.4)$$

We note that (4.4) does not restrict the domain of u_0 , but this will not be the case when temperature is turned on. Since $\lambda < 1$, a baryon configuration exists if $a > a_<$, $a_< = 5/8 = 0.625$. This value was found in [3, 4]. Using (4.2) and (4.4) we can find the binding energy in terms of the physical length

$$E = -\frac{R^2}{2\pi L} \frac{k\sqrt{1-\lambda^2}}{3} \left(\mathcal{J} - \frac{5-4a}{4a} \right) \mathcal{I}. \quad (4.5)$$

From this expression it is apparent why the possibility $a = 5/8$ ($\lambda = 1$) is discarded. The energy of the baryon goes as $1/L$ as it was expected by conformal invariance and has the familiar for these type of computations dependence on the 't Hooft coupling.

4.1.2 Stability analysis

Next we consider the stability behavior of the conformal case. For the longitudinal fluctuations we have to solve (3.12), where

$$\delta r(u) = A \int_u^\infty du \frac{u^2}{(u^4 - u_1^4)^{3/2}} = \frac{A}{3u^3} {}_2F_1 \left(\frac{3}{4}, \frac{3}{2}, \frac{7}{4}; \frac{u_1^4}{u^4} \right). \quad (4.6)$$

Substituting (4.4) and (4.6) in (3.12) we find the following transcendental equation for the zero mode to exist

$${}_2F_1 \left(\frac{3}{4}, \frac{3}{2}, \frac{7}{4}; 1 - \lambda^2 \right) = \frac{3}{2\lambda(1 + \lambda^2)}. \quad (4.7)$$

Using (4.4) and (4.7) we find numerically a critical value for a , i.e. $a_c \simeq 0.813$.

Thus, the stability analysis sets a low bound for a which is still less than unity. However, given that this corresponds to the conformal point of a maximally $\mathcal{N} = 4$ supersymmetric theory we find it remarkable. The only other case of perturbative instability in the conformal point is for the quark-antiquark potential in the β -deformed $\mathcal{N} = 1$ theory [16], whose supergravity dual was constructed in [17]. In this case the instability occurs for values of the σ -deformation parameter larger than a critical value [7].

4.2 Non-extremal D3-branes

4.2.1 Classical solution

In this section we consider a stack of N non-extremal $D3$ -branes [13]. In the field theory limit the metric has the form of (2.1) where

$$G_{tt} = -G_{uu}^{-1} = -\frac{u^2}{R^2} \left(1 - \frac{\mu^4}{u^4}\right), \quad G_{xx} = \frac{u^2}{R^2}. \quad (4.8)$$

The corresponding Hawking temperature is $T_H = \mu/(\pi R^2)$. In what follows, we will compute in units of μ , so that we set $\mu = 1$. The radius and the energy read

$$\begin{aligned} L &= R^2 \frac{\sqrt{u_1^4 - 1}}{3u_0^3} F_1 \left(\frac{3}{4}, \frac{1}{2}, \frac{1}{2}, \frac{7}{4}; \frac{1}{u_0^4}, \frac{u_1^4}{u_0^4} \right), \\ E &= \frac{k}{2\pi} \left\{ \mathcal{E} + \frac{u_0}{4a} \sqrt{1 - \frac{1}{u_0^4}} + \frac{1-a}{a} (u_0 - 1) \right\}, \\ \mathcal{E} &= -u_0 F_1 \left(-\frac{1}{4}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{4}; \frac{1}{u_0^4}, \frac{u_1^4}{u_0^4} \right) + 1, \end{aligned} \quad (4.9)$$

where $F_1(a, b_1, b_2, c; z_1, z_2)$ is the Appell hypergeometric function and $u_0 \geq u_1 \geq 1$. From (2.17) we find that in this case the no-force condition on the u -axis yields

$$\cos \Theta = \frac{1-a}{a} + \frac{1+1/u_0^4}{4a\sqrt{1-1/u_0^4}}, \quad \cos \Theta = \sqrt{\frac{u_0^4 - u_1^4}{u_0^4 - 1}}. \quad (4.10)$$

Employing the fact that $\cos \Theta \leq 1$ in (4.10) we end up with a second order polynomial inequality which has a solution for $a \geq 5/8$, namely

$$\begin{aligned} u_0 &\geq u_<(a), \quad u_<(a) = 1/\lambda^{1/4}, \\ \lambda &= -9 - 32a(a-1) + 4\sqrt{2}(2a-1)\sqrt{3-8a(1-a)}, \\ a &\in [5/8, 1], \quad \lambda \in [0, -9 + 4\sqrt{6}], \quad u_0 \in [\infty, u_<(1)], \quad u_<(1) \simeq 1.058. \end{aligned} \quad (4.11)$$

However, if $a = 5/8$ there is no classical solution since $u_0 \rightarrow \infty$. For $a > 5/8$ the solution of (4.10) is

$$u_1 = u_0(1 - \Lambda^2(u_0))^{1/4}, \quad \Lambda(u_0) = \frac{1-a}{a} \sqrt{1 - 1/u_0^4} + \frac{1+1/u_0^4}{4a}. \quad (4.12)$$

In this case we can not express E in terms of L . In fact E is a doubled-valued function of L with energetically favorable and unfavorable branches, as depicted in figure 2a. It is similar to the quark-antiquark potential for the finite temperature case [14].

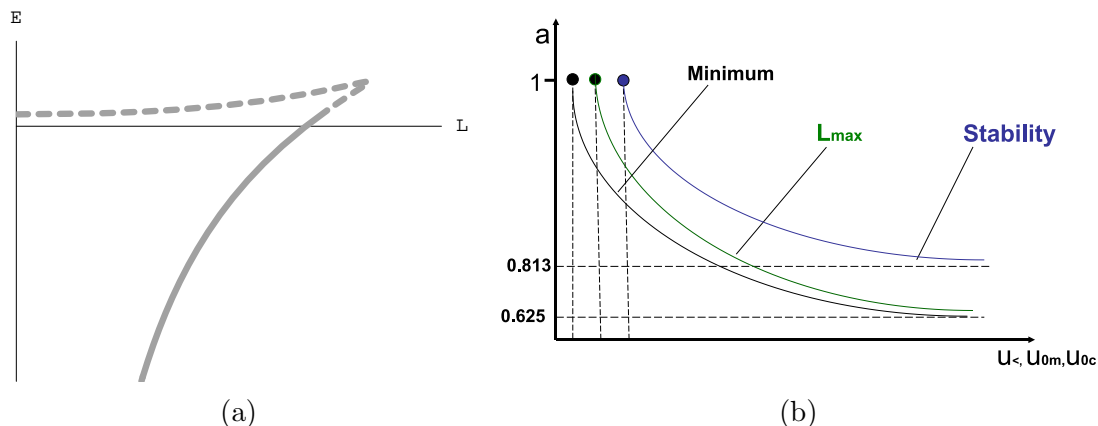


Figure 2: (a) Plot of $E(L)$ for non extremal $D3$ -branes. The various types of line correspond to stable (solid grey) and unstable (dashed grey) configurations. As a decreases unstable part extends to the lower branch as well. This behavior is described in section (4.2.2). (b) From the left to the right: i) Plot of $u_<$ versus a , ii) Plot a versus u_{0m} , iii) Plot of u_{0c} versus a . The regions on the right and left hand side of the stability line are stable and unstable, respectively.

4.2.2 Stability analysis

Next we will consider the stability behavior of the non-extremal $D3$ -branes. For the longitudinal fluctuations we have to solve (3.12) where

$$\delta r(u) = A \int_u^\infty du \frac{\sqrt{u^4 - 1}}{(u^4 - u_1^4)^{3/2}} = \frac{A}{3u^3} F_1 \left(\frac{3}{4}, -\frac{1}{2}, \frac{3}{2}, \frac{7}{4}, \frac{1}{u^4}, \frac{u_1^4}{u^4} \right). \quad (4.13)$$

From (3.12) and (4.13) we find that the zero mode condition is given by the transcendental equation

$$\frac{3}{2\sqrt{u_0^4 - u_1^4}} = \frac{2u_0^4 - u_1^4 - 1}{(u_0^4 - 1)^{3/2}} F_1 \left(\frac{3}{4}, -\frac{1}{2}, \frac{3}{2}, \frac{7}{4}, \frac{1}{u_0^4}, \frac{u_1^4}{u_0^4} \right). \quad (4.14)$$

From numerical considerations we find that for every value of a larger than the critical value found in the conformal case $a_c \simeq 0.813$ there is a zero mode on the energetically favored branch of $E(L)$ i.e. $u_{0c} > u_{0m}$. Its solution has the expansion

$$a_c \simeq 0.813 + \frac{0.679}{u_{0c}^4} + \frac{0.778}{u_{0c}^8} + \mathcal{O} \left(\frac{1}{u_{0c}^{12}} \right). \quad (4.15)$$

As a_c decreases u_{0c} increases and for $a_c \simeq 0.813$ even the conformal part becomes unstable. The results of our stability analysis are summarized in figure2b.

Also, for the particular case of $a = 1$ we present table 1.

5. Magnetic flux and stability of the baryon vertex

In this section we will generalize our discussion to the case where we turn on, besides the electric ones, non-trivial magnetic flux components in F_{ij} , as in [12]. The fact that this is possible could be seen, from the fact that the S^5 -sphere can be regarded as a $U(1)$

Values for $a = 1$	Minimum	Critical	Maximum
u_0	1.0581	1.4292	1.3437
L/R^2	0	0.2920	0.2951
E/N	0.0097	0.0266	0.0273

Table 1: The length and energy for the minimum, critical and maximum values of u_0 for $a = 1$.

fibre over \mathbb{CP}^2 with a non-trivial fibre connection [19]. This U(1) connection, introduces a generalization of the baryon vertex in which non-zero magnetic components of the Born-Infeld field F_{ij} appear in the \mathbb{CP}^2 directions in such a way that it is self-dual with respect to that metric. For topological reasons, its flux over non-trivial two-cycles of \mathbb{CP}^2 is quantized and n^2 represents the instanton number. Working along similar lines as in section (2.2) of [12], we find that the action of the $D5$ -brane (2.5) in our conventions gets modified to

$$S_{D5} = \frac{TN R \sqrt{-G_{tt}}}{8\pi} \nu, \tag{5.1}$$

where

$$\nu \equiv 1 + 32\pi^2 b, \quad b \equiv \frac{n^2}{R^4}. \tag{5.2}$$

Note that the relevant parameter is not the instanton number n^2 but its ratio with the 't Hooft coupling. The analogous expression to (2.15) for the binding energy of the baryon is

$$E = \frac{k}{2\pi} \left\{ \int_{u_0}^{\infty} du \sqrt{F} - \int_{u_{\min}}^{\infty} du \sqrt{g} + \frac{1-a}{a} \int_{u_{\min}}^{u_0} du \sqrt{g} + \nu \frac{R \sqrt{-G_{tt}}}{4a} \Big|_{u=u_0} \right\}, \tag{5.3}$$

while the expression for the radius of the baryon is given by the same equation as in (2.12). The analogous expression to (2.17) for the no-force condition on the u -axis yields¹

$$\begin{aligned} \cos \Theta &= \frac{1-a}{a} + \nu R \frac{\partial_u \sqrt{-G_{tt}}}{4a \sqrt{g}} \Big|_{u=u_0}, \\ \cos \Theta &\equiv \sqrt{1 - f_1/f_0}. \end{aligned} \tag{5.4}$$

The stability analysis of this classical solution is easy to discuss. Since the Born-Infeld field is self-dual, the determinant under the square root in the action of the $D5$ -brane (2.5) is a perfect square, and there no mixing terms of the induced metric of the $D5$ -brane and the Born-Infeld field. Furthermore, the Born-Infeld field is a two-form on \mathbb{CP}^2 that remains unchanged since perturbing the position of the $D5$ -brane we keep the gauge choice (2.13). Hence, the stability analysis described in section 3, is applicable for non-zero magnetic flux as well.

We will next specialize to the conformal and non-extremal D3-branes cases.

¹The conditions (3.13) and (3.15) are also valid in this case as well. In using (3.14) one should include an overall factor of ν to the second term in the right hand side.

5.1 The conformal case

We will first consider the classical solution for the conformal case. The whole treatment can be done in terms of an effective parameter a_{eff} (defined below) and using our previous results in the absence of magnetic charge. The analogous expression to (4.4) reads

$$u_1 = u_0(1 - \lambda_{\text{eff}}^2)^{1/4}, \quad \lambda_{\text{eff}} = \frac{5 - 4a_{\text{eff}}}{4a_{\text{eff}}}, \quad a_{\text{eff}} \equiv \frac{a}{1 + \frac{32\pi^2 b}{5}}. \quad (5.5)$$

Since $\lambda_{\text{eff}} < 1$ a baryon configuration exists for

$$a_{\text{eff}} > 5/8 \implies a > a_<, \quad a_< = \frac{5}{8} + 4\pi^2 b. \quad (5.6)$$

However, since $a \leq 1$ we get a bound for the instanton number, namely that

$$a_< < 1 \implies \frac{n^2}{R^4} < \frac{3}{32\pi^2} \simeq 0.0095, \quad (5.7)$$

which is the bound found in [12] for a classical configuration to exist. The binding energy in terms of the radius reads

$$E = -\frac{R^2}{2\pi L} \frac{k\sqrt{1 - \lambda_{\text{eff}}^2}}{3} \left(\mathcal{J} - \frac{5 - 4a_{\text{eff}}}{4a_{\text{eff}}} \right) \mathcal{I}, \quad (5.8)$$

where \mathcal{I} and \mathcal{J} were given in (4.3). We have shown in section 4 that there is a zero mode for $a_{\text{eff}} \simeq 0.813$, for which the system becomes unstable. This improves the above bound for the instanton number, in comparison to the 't Hooft coupling, to

$$\frac{n^2}{R^4} \lesssim 0.0036, \quad (5.9)$$

that should be respected for the classical configuration not only to exist, but also to be perturbatively stable. Stability analysis sets lower and higher bounds for the instanton number and a , 0.0036 and 0.813 respectively. This behavior is depicted in figure3.

5.2 Non-extremal D3-branes

Finally, we will consider the non-extremal D3-branes. The analogous expression to (4.9) for the energy read

$$E = \frac{k}{2\pi} \left\{ \mathcal{E} + \frac{1-a}{a}(u_0 - 1) + \nu \frac{u_0}{4a} \sqrt{1 - \frac{1}{u_0^4}} \right\},$$

$$\mathcal{E} = -u_0 F_1 \left(-\frac{1}{4}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{4}; \frac{1}{u_0^4}, \frac{u_1^4}{u_0^4} \right) + 1. \quad (5.10)$$

The analogous expression to (4.10) no-force condition, has a solution for $a > \frac{4 + \nu}{8}$, namely that

$$u_1 = u_0(1 - \Lambda^2(u_0))^{1/4}, \quad \Lambda(u_0) = \frac{1-a}{a} \sqrt{1 - 1/u_0^4} + \nu \frac{1 + 1/u_0^4}{4a},$$

$$u_0 \geq u_<, \quad u_< = 1/\lambda^{1/4}, \quad (5.11)$$

$$\lambda = \frac{1}{\nu^2} \left(-8 - \nu^2 + 32(-a + 1)a + 4\sqrt{2}(2a - 1)\sqrt{2 - 8a(1 - a) + \nu^2} \right).$$

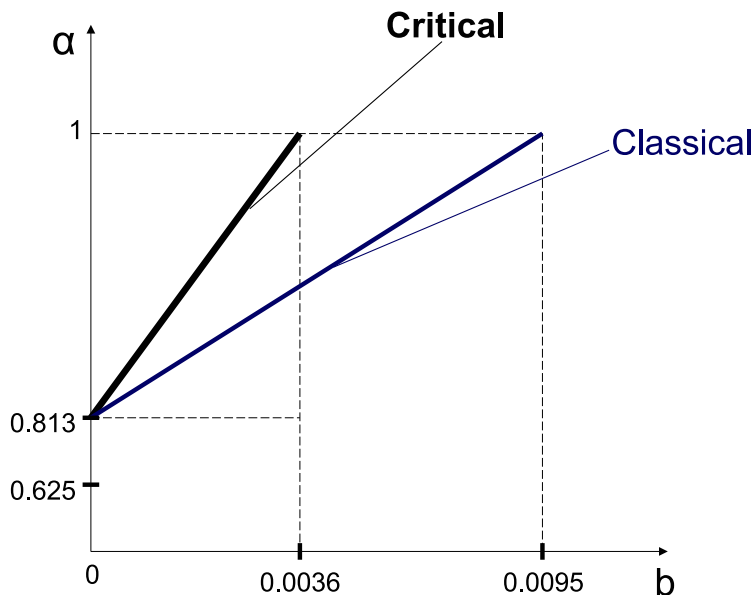


Figure 3: Classical and critical lines of a versus b , i.e. $a_{\text{eff}} = 5/8, 0.813$, respectively. There is a classical solution for values of (a, b) , on the right hand side of the classical line. As for stability, the regions on the left and the right hand sides of the critical line are stable and unstable, respectively.

The energy versus radius is a doubled valued function with an energetically favorable and an energetically unfavorable branch, as depicted in figure2a.

Turning now to the stability behavior of the classical solution, we found that for every value of a and instanton number above the critical line of figure3, there is always a zero-mode on the energetically favorable branch of $E(L)$, giving rise to a critical curve bounded from below by the conformal critical line. As a and/or the instanton number reach the critical line, even the conformal part of the diagram becomes unstable.

6. Summary and discussion

In this paper we first constructed classical configurations with strings and a $D5$ -brane that represent bound, baryon-like states, within the gauge/gravity correspondence and for a general class of backgrounds. We worked out explicit examples in the $\mathcal{N} = 4$ SYM for $SU(N)$ at the conformal point and at finite temperature. Stability restricts the value of k to be larger than a critical value even in the conformal case ($0.813N$, higher than the classical lower bound $0.625N$). In addition, a part of the energetically favorable branch of the multi-branch potential arising in the finite temperature case should be disregarded as unstable, even if $k = N$. If the baryon vertex is supplemented by magnetic charge, we showed that stability requires an upper bound for the instanton number, as compared to the 't Hooft coupling, associated with a self-dual gauge field. This is unlike the findings for quark-antiquark potentials in [6, 7] where stability discards energetically unfavorable branches completely and for the dyon-dyon potentials in [8], where parts of the energetically unfavorable branch become perturbatively stable.

One may well wonder how generic is our discussion in the finite temperature case, having used a specific black $D3$ -brane background. We have checked that for the Rindler space, that represents the horizon of any black hole-type solution similar phenomena occur. There is a critical value of k below which there is instability and also part of the energetically favorable branch is unstable. Hence, our results are quite generic for a large class of backgrounds with black hole-type behavior. Moreover, our techniques could be extendable to theories with dynamical flavor, such as the one in [20].

Our work is also straightforwardly extendable to supergravity backgrounds in which off-diagonal metric elements appear. This covers cases with non-vanishing Wess-Zumino term and cases of moving baryons in hot quark-gluon plasmas. The classical solution of a moving baryon in hot plasma was studied in [15]. The end result is that for a uniform circle distribution at the boundary of AdS_5 , the binding energy versus length has two branches and for high enough velocities $L_{\max} \sim (1 - v^2)^{1/4}/T_H$, where the proportionality coefficient depends on the direction that the baryon moves, with respect to the axis of the circle. We have checked the stability behavior and found that for perpendicular to the circle distribution motion, there is an instability on the energetically favored branch and as speed increases and/or a decreases, even the conformal part of the diagram becomes unstable. We expect this behavior to be valid for arbitrary quark distributions in baryons and moving directions. It is quite interesting if this is further pursued.

One could wonder if there are cases when any number of quarks below $k = N$ gives rise to unstable configurations, thus realizing our original motivation that prompted this work, i.e. only colorless states can exist. It is not a surprise that this does not seem to happen for the $\mathcal{N} = 4$ SYM theory, since after all this is a maximally supersymmetric finite theory. It will be interesting to explore this issue in other theories with reduced supersymmetry.

In our investigation we have chosen to work for simplicity with a quark distribution in the baryon that is homogeneous and confined on a spherical shell. This choice is quite arbitrary and although it preserves radial symmetry it is not a priori the one that the quarks would prefer if, for instance, we apply a variational principle that minimizes the total energy of the configuration. It would be very interesting to find the mechanism that determines from first principles the appropriate quark distribution in a baryon within this framework. As a last comment on that, we have checked that a randomly chosen radially symmetric distribution of quarks in the baryon is inappropriate since in that case, not even the classical configuration exists, i.e. the no-force condition at the baryon vertex is not satisfied.

Acknowledgments

We acknowledge support provided through the European Community's program "Constituents, Fundamental Forces and Symmetries of the Universe" with contract MRTN-CT-2004-005104 and a program with contract ANR-05-BLAN-0079-02.

In addition, K. Sfetsos acknowledges support provided by the Groupemeur d'Intérêt scientifique P2I (physique des deux infinis).

In addition, K. Siampos acknowledges support provided by the Greek State Scholarship Foundation (IKY). Also would like to thank the department of Mathematics of Kings

College London for hospitality and support provided by the “European Superstring Theory Network” with contract MCFH-2004-512194, where part of this work was done.

References

- [1] J.M. Maldacena, *The large- N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [[hep-th/9711200](#)]; S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [[hep-th/9802109](#)]; E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)]; *Anti-de Sitter space, thermal phase transition and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505 [[hep-th/9803131](#)].
- [2] E. Witten, *Baryons and branes in anti de Sitter space*, *JHEP* **07** (1998) 006 [[hep-th/9805112](#)]; D.J. Gross and H. Ooguri, *Aspects of large N gauge theory dynamics as seen by string theory*, *Phys. Rev.* **D 58** (1998) 106002 [[hep-th/9805129](#)].
- [3] A. Brandhuber, N. Itzhaki, J. Sonnenschein and S. Yankielowicz, *Baryons from supergravity*, *JHEP* **07** (1998) 020 [[hep-th/9806158](#)].
- [4] Y. Imamura, *Baryon mass and phase transitions in large- N gauge theory*, *Prog. Theor. Phys.* **100** (1998) 1263 [[hep-th/9806162](#)].
- [5] J.M. Maldacena, *Wilson loops in large- N field theories*, *Phys. Rev. Lett.* **80** (1998) 4859 [[hep-th/9803002](#)]; S.-J. Rey and J.-T. Yee, *Macroscopic strings as heavy quarks in large- N gauge theory and anti-de Sitter supergravity*, *Eur. Phys. J.* **C 22** (2001) 379 [[hep-th/9803001](#)].
- [6] S.D. Avramis, K. Sfetsos and K. Siampos, *Stability of strings dual to flux tubes between static quarks in $N = 4$ SYM*, *Nucl. Phys.* **B 769** (2007) 44 [[hep-th/0612139](#)].
- [7] S.D. Avramis, K. Sfetsos and K. Siampos, *Stability of string configurations dual to quarkonium states in AdS/CFT*, *Nucl. Phys.* **B 793** (2008) 1 [[arXiv:0706.2655](#)].
- [8] K. Sfetsos and K. Siampos, *String junctions in curved backgrounds, their stability and dyon interactions in SYM*, *Nucl. Phys.* **B 797** (2008) 268 [[arXiv:0710.3162](#)].
- [9] A. Brandhuber and K. Sfetsos, *Wilson loops from multicentre and rotating branes, mass gaps and phase structure in gauge theories*, *Adv. Theor. Math. Phys.* **3** (1999) 851 [[hep-th/9906201](#)].
- [10] Y. Imamura, *Supersymmetries and BPS configurations on Anti-de Sitter space*, *Nucl. Phys.* **B 537** (1999) 184 [[hep-th/9807179](#)].
- [11] C.G. Callan Jr., A. Guijosa and K.G. Savvidy, *Baryons and string creation from the fivebrane worldvolume action*, *Nucl. Phys.* **B 547** (1999) 127 [[hep-th/9810092](#)].
- [12] B. Janssen, Y. Lozano and D. Rodriguez-Gomez, *The baryon vertex with magnetic flux*, *JHEP* **11** (2006) 082 [[hep-th/0606264](#)].
- [13] G.T. Horowitz and A. Strominger, *Black strings and P-branes*, *Nucl. Phys.* **B 360** (1991) 197.

- [14] S.-J. Rey, S. Theisen and J.-T. Yee, *Wilson-Polyakov loop at finite temperature in large- N gauge theory and anti-de Sitter supergravity*, *Nucl. Phys. B* **527** (1998) 171 [[hep-th/9803135](#)];
A. Brandhuber, N. Itzhaki, J. Sonnenschein and S. Yankielowicz, *Wilson loops in the large- N limit at finite temperature*, *Phys. Lett. B* **434** (1998) 36 [[hep-th/9803137](#)].
- [15] C. Athanasiou, H. Liu and K. Rajagopal, *Velocity dependence of baryon screening in a hot strongly coupled plasma*, *JHEP* **05** (2008) 083 [[arXiv:0801.1117](#)].
- [16] R.G. Leigh and M.J. Strassler, *Exactly marginal operators and duality in four-dimensional $N = 1$ supersymmetric gauge theory*, *Nucl. Phys. B* **447** (1995) 95 [[hep-th/9503121](#)].
- [17] O. Lunin and J.M. Maldacena, *Deforming field theories with $U(1) \times U(1)$ global symmetry and their gravity duals*, *JHEP* **05** (2005) 033 [[hep-th/0502086](#)].
- [18] B. Baumgartner, H. Grosse and A. Martin, *Order of levels in potential models*, *Nucl. Phys. B* **254** (1985) 528;
C. Bachas, *Convexity of the quarkonium potential*, *Phys. Rev. D* **33** (1986) 2723.
- [19] C.N. Pope, *Eigenfunctions and Spin (c) structures in CP^2* , *Phys. Lett. B* **97** (1980) 417.
- [20] R. Casero, C. Núñez and A. Paredes, *Towards the string dual of $N = 1$ SQCD-like theories*, *Phys. Rev. D* **73** (2006) 086005 [[hep-th/0602027](#)].